

**FINITE p -GROUPS ALL OF WHOSE PROPER SUBGROUPS
HAVE ITS DERIVED SUBGROUP OF ORDER AT MOST p**

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ABSTRACT. We give in Theorem 7 a complete characterization of the title groups.

Here we give a complete characterization of the title groups. This result is important for the structure theory of finite p -groups and also it solves the Problem 39 stated by Y. Berkovich in [1]. In the proofs we use partly some ideas of J. Q. Zhang and X. H. Li ([5, Proposition 3]) and V. Čepulić and O. Pylyavska ([4, Proposition 5]). To facilitate the proof of the main result (Theorem 7), we shall first prove some auxiliary results.

Our notation is standard (see [1]) and we consider here only finite p -groups.

PROPOSITION 1. *Let G be a title group. Then for all $x, y \in G$ such that $\langle x, y \rangle < G$ we have $\text{o}([x, y]) \leq p$ and $[x, y] \in Z(G)$.*

PROOF. Suppose that $[x, y] \neq 1$. Let X be a maximal subgroup of G containing $\langle x, y \rangle$. Then $X' = \langle [x, y] \rangle \trianglelefteq G$ with $\text{o}([x, y]) = p$ and so $[x, y] \in Z(G)$. \square

PROPOSITION 2. *If G is a title group, then G' is abelian of order $\leq p^3$.*

PROOF. We may assume that G is nonabelian. Let $X \neq Y$ be two maximal subgroups of G . Then $|X'| \leq p$ and $|Y'| \leq p$. By a result of A. Mann (Exercise 1.69(a) in [1]), $|G' : (X'Y')| \leq p$ and so $|G'| \leq p^3$. If G' would be nonabelian, then $|G'| = p^3$ and $Z(G')$ (being of order p) is cyclic and so (by an elementary result of W. Burnside, see Lemma 1.4 in [1]) G' is also cyclic, a contradiction. Hence G' is abelian. \square

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PROPOSITION 3 (Zhang and Li). *If G is a title group and $|G'| \geq p^2$, then $d(G) \leq 3$.*

PROOF. Assume that $|G'| \geq p^2$. Then G is not minimal nonabelian and so there exists a maximal subgroup A with $|A'| = p$ and we have $A' \triangleleft G$. Suppose that $M' \leq A'$ for each maximal subgroup M of G . Then G/A' is minimal nonabelian. But then $d(G/A') = 2$, $A' \leq \Phi(G)$ and so $d(G) = 2$ and we are done in this case.

We may assume that G has a maximal subgroup B such that $B' \not\leq A'$. We get $|A'| = |B'| = p$ and $A' \cap B' = \{1\}$. Let $a_1, a_2 \in A$ and $a_3, a_4 \in B$ be such that $A' = \langle [a_1, a_2] \rangle$ and $B' = \langle [a_3, a_4] \rangle$. Since $|\langle a_1, a_2, a_3, a_4 \rangle'| \geq p^2$, we get $\langle a_1, a_2, a_3, a_4 \rangle = G$ and so $d(G) \leq 4$.

We assume, by a way of contradiction, that $d(G) = 4$. By Proposition 1, for any $x, y \in G$ we have $o([x, y]) \leq p$ and $[x, y] \in Z(G)$. This implies that G' is elementary abelian and $G' \leq Z(G)$. In particular, G is of class 2.

For any $k \in \{1, 2\}$ and $l \in \{3, 4\}$, we have $\langle a_1, a_2, a_l \rangle < G$ and $\langle a_k, a_3, a_4 \rangle < G$ and so $\langle a_1, a_2, a_l \rangle' = \langle [a_1, a_2] \rangle$ and $\langle a_k, a_3, a_4 \rangle' = \langle [a_3, a_4] \rangle$. It follows that

$$[a_k, a_l] \in \langle [a_1, a_2] \rangle \cap \langle [a_3, a_4] \rangle = \{1\}.$$

This implies

$$[a_1, a_2 a_3] = [a_1, a_2][a_1, a_3] = [a_1, a_2] \text{ and } [a_2 a_3, a_4] = [a_2, a_4][a_3, a_4] = [a_3, a_4].$$

But then $\langle a_1, a_2 a_3, a_4 \rangle$ is a proper subgroup of G and we have $|\langle a_1, a_2 a_3, a_4 \rangle'| \geq p^2$, a contradiction. Our proposition is proved. \square

PROPOSITION 4 (Y. Berkovich). *Suppose that G is a nonabelian p -group. If $d(G) = 2$, then $H' < G'$ for each $H < G$.*

PROOF. Let $R < G'$ be a G -invariant subgroup of index p in G' . Then $|(G/R)'| = p$ and $d(G/R) = 2$. This implies that G/R is minimal nonabelian. For each maximal subgroup H of G , $H' \leq R < G'$ and we are done. \square

PROPOSITION 5 (Ćepulić and Pylyavska). *Let G be a title p -group with $p > 2$. Then for any $a, b \in G$, we have $[a^p, b] = [a, b^p] = [a, b]^p$.*

PROOF. We set $g = [a, b]$. If g commutes with a , then for each $n \geq 1$ we prove by induction that $[a^n, b] = [a, b]^n$. Indeed, for $n > 1$,

$$\begin{aligned} [a^n, b] &= [aa^{n-1}, b] = [a, b]^{a^{n-1}} [a^{n-1}, b] \\ &= [a, b][a^{n-1}, b] = [a, b][a, b]^{n-1} = [a, b]^n. \end{aligned}$$

In particular, we have $[a^p, b] = [a, b]^p$.

We assume now that $[g, a] = z \neq 1$. Since $\langle g, a \rangle < G$, Proposition 1 implies that $o(z) = p$ and $z \in Z(G)$. We note that

$$g^a = a^{-1}ga = g(g^{-1}a^{-1}ga) = g[g, a] = gz \text{ and so } g^{a^i} = gz^i$$

for all $i \geq 1$. We have

$$\begin{aligned} [a^p, b] &= [a \cdot a^{p-1}, b] = [a, b]^{a^{p-1}} [a^{p-1}, b] = [a, b]^{a^{p-1}} [a \cdot a^{p-2}, b] \\ &= [a, b]^{a^{p-1}} [a, b]^{a^{p-2}} [a^{p-2}, b] \end{aligned}$$

and so continuing we get finally:

$$\begin{aligned} [a^p, b] &= [a, b]^{a^{p-1}} [a, b]^{a^{p-2}} \dots [a, b]^a [a, b] \\ &= g^{a^{p-1}} g^{a^{p-2}} \dots g^a g = (gz^{p-1})(gz^{p-2}) \dots (gz)g \\ &= g^p z^{(p-1)+(p-2)+\dots+1} = g^p z^{(p-1)\frac{p}{2}} = g^p = [a, b]^p, \end{aligned}$$

where we have used the fact that $p > 2$. We have proved that in any case we get $[a^p, b] = [a, b]^p$. Now, $[a, b^p] = [b^p, a]^{-1} = [b, a]^{-p} = [a, b]^p$, and so our proposition is proved. \square

PROPOSITION 6. *Suppose that G is a p -group which has one of the following properties:*

- (a) $|G'| \leq p$;
- (b) $d(G) = 2$, $|G'| = p^2$;
- (c) $p > 2$, $d(G) = 2$, $\text{cl}(G) = 3$, $G' \cong E_{p^3}$, $\mathcal{U}_1(G) \leq Z(G)$;
- (d) $d(G) = 3$, $\text{cl}(G) = 2$, $G' \cong E_{p^3}$ or E_{p^2} .

Then G has the title property, i.e., $|H'| \leq p$ for each proper subgroup H of G .

PROOF. If G is a p -group in (a), then obviously G has the title property. Suppose that G is a p -group in (b). By Proposition 4, for each $H < G$ we have $H' < G'$ and so G has the title property.

Now assume that G is a p -group in (c). Since $\text{cl}(G) = 3$, we have $\{1\} \neq K_3(G) \leq Z(G)$. But $d(G) = 2$ and so $\{1\} \neq G'/K_3(G)$ is cyclic and therefore $K_3(G) \cong E_{p^2}$. We have $\mathcal{U}_1(G) \leq Z(G)$ and so $\Phi(G) = \mathcal{U}_1(G)G'$ is abelian and $G/\Phi(G) \cong E_{p^2}$. Also, $\mathcal{U}_1(G)K_3(G) \leq Z(G)$ and in fact $\mathcal{U}_1(G)K_3(G) = Z(G)$. Indeed, if $\mathcal{U}_1(G)K_3(G) < Z(G)$, then $G/Z(G) \cong E_{p^2}$. But in that case G has $p+1$ abelian maximal subgroups and this implies (Exercise P1 in [3]) $|G'| = p$, a contradiction. Let M be any maximal subgroup of G so that $|M : \Phi(G)| = p$. Then M is either abelian or $Z(G) = Z(M)$ and $M/Z(M) \cong E_{p^2}$. In the second case we may use Lemma 1.1 in [1] since $\Phi(G)$ is an abelian maximal subgroup of M . From $|M| = p|Z(M)||M'|$, we get $|M'| = p$. We have proved that in this case G has the title property.

Suppose that G is a p -group in (d). For any $x, y \in G$ we have $[x^p, y] = [x, y]^p = 1$ and so $\mathcal{U}_1(G) \leq Z(G)$. It follows that $\Phi(G) = \mathcal{U}_1(G)G' \leq Z(G)$ and $G/\Phi(G) \cong E_{p^3}$. Let M be any maximal subgroup of G so that $M/\Phi(G) \cong E_{p^2}$. It follows that $p+1$ maximal subgroups of M which contain $\Phi(G)$ are abelian. This implies that $|M'| \leq p$ and we are done. \square

THEOREM 7. *A p -group G has the property that each proper subgroup of G has its derived subgroup of order at most p if and only if one of the following holds:*

- (a) $|G'| \leq p$;
- (b) $d(G) = 2$, $|G'| = p^2$;
- (c) $p > 2$, $d(G) = 2$, $\text{cl}(G) = 3$, $G' \cong E_{p^3}$, $\mathcal{U}_1(G) \leq Z(G)$
(note that such p -groups exist. See for example A_2 -groups of order p^5 , $p > 2$, in Proposition 71.5(b) in [2]);
- (d) $d(G) = 3$, $\text{cl}(G) = 2$, $G' \cong E_{p^3}$ or E_{p^2} . Here we have $\Phi(G) = Z(G)$.

PROOF. If G is a p -group in (a), (b), (c) or (d), then Proposition 6 implies $|H'| \leq p$ for each subgroup $H < G$.

Suppose that G is a p -group all of whose proper subgroups have its derived subgroup of order $\leq p$. If $|G'| \leq p$, then we have the groups in part (a) of our theorem. In what follows we assume that $|G'| \geq p^2$. By Proposition 2, G' is abelian of order p^2 or p^3 . By Proposition 3, we have $d(G) \leq 3$.

(i) First assume $d(G) = 2$. If $|G'| = p^2$, then we have obtained the groups in part (b) of our theorem. In the sequel we shall assume here $|G'| = p^3$. By a result of A. Mann (Exercise 1.69(a) in [1]), all $p+1$ maximal subgroups M_i ($i = 1, 2, \dots, p+1$) of G are nonabelian, $|M'_i| = p$ and for any $i \neq j$ we have $M'_i \cap M'_j = \{1\}$ so that $M'_i \times M'_j \cong E_{p^2}$ and $M'_i \times M'_j \leq Z(G)$. If $\text{cl}(G) = 2$, then $d(G) = 2$ would imply that G' is cyclic, contrary to the existence of the subgroup $M'_i \times M'_j \cong E_{p^2}$. Hence $\text{cl}(G) \geq 3$. But $\{1\} \neq K_3(G) = [G, G'] \leq M'_i \times M'_j \leq Z(G)$ and so $\text{cl}(G) = 3$. We set $E = M'_i \times M'_j = G' \cap Z(G) \cong E_{p^2}$. Whenever $a, b \in G$ are such that $\langle a, b \rangle = G$, then $[a, b] \in G' - E$. Indeed, if $1 \neq [a, b] \in E$, then $\text{o}([a, b]) = p$ and $[a, b] \in Z(G)$. But then $G/\langle [a, b] \rangle$ is abelian and so $G' = \langle [a, b] \rangle$ is of order p , a contradiction. Let $g = [a, b]$, where $\langle a, b \rangle = G$ and $g \in G' - E$. For any $x \in G$ we have $g^x = ge$ with some $e \in E$. Then $g^{x^i} = ge^i$ and so $g^{x^p} = g$. It follows that $\mathcal{U}_1(G)$ centralizes G' and so $\Phi(G) = \mathcal{U}_1(G)G'$ centralizes G' .

(i1) Now assume $p > 2$. Suppose in addition that G' is not elementary abelian. Then $E = \Omega_1(G')$ and set $\{1\} \neq \mathcal{U}_1(G') = \langle s \rangle < E$ so that $G'/\langle s \rangle \cong E_{p^2}$. If $K_3(G) = [G, G'] = \langle s \rangle$, then $G/\langle s \rangle$ is of class 2 so that $d(G/\langle s \rangle) = 2$ would imply that $G'/\langle s \rangle = (G/\langle s \rangle)'$ is cyclic, a contradiction. Hence there is an element $c \in G - \Phi(G)$ such that $g^c = gl$ with $l = [g, c] \in E - \langle s \rangle$. Let $d \in G - \Phi(G)$ be such that $\langle c, d \rangle = G$ so that $[c, d] \in G' - E$. By Proposition 5, $[c, d^p] = [c, d]^p = s^j$, where $j \not\equiv 0 \pmod{p}$. Consider the maximal subgroup $C = \langle \Phi(G), c \rangle$. Since $g, c, d^p \in C$, we have $C' \geq \langle [g, c], [c, d^p] \rangle = \langle l, s^j \rangle = E \cong E_{p^2}$, a contradiction. We have proved that $G' \cong E_{p^3}$. For any $x, y \in G$ we get by Proposition 5, $[x^p, y] = [x, y]^p = 1$ and so $\mathcal{U}_1(G) \leq Z(G)$. We have obtained the groups given in part (c) of our theorem.

(i2) It remains to consider the case $p = 2$. Assume in addition that $\{1\} \neq K_3(G) = [G, G'] < E$ and set $[G, G'] = \langle u \rangle$, where u is an involution

in $E \leq Z(G)$. Note that $\Phi(G)$ centralizes G' and for each $x \in G - \Phi(G)$ and $y \in G' - E$ we have $y^x = yu'$ with $u' \in \langle u \rangle$. Set $G_0 = C_G(G')$ so that we have $|G : G_0| = |G_0 : \Phi(G)| = 2$. Since $G/\langle u \rangle$ is of class 2 and $d(G/\langle u \rangle) = 2$, we have $G'/\langle u \rangle$ is cyclic. Hence if $g \in G' - E$, then $g^2 = v$ is an involution in $E - \langle u \rangle$ and therefore $E = \Omega_1(G') = \langle u, v \rangle$ and $\mathcal{U}_1(G') = \langle v \rangle$. Take some elements $a \in G_0 - \Phi(G)$ and $b \in G - G_0$. Then $\langle a, b \rangle = G$ and therefore $[a, b] = h \in G' - E$ with $h^2 = v$, $h^a = h$ and $h^b = hu$. Consider the maximal subgroup $H = \langle \Phi(G), b \rangle$. Since

$$[a^2, b] = [a, b]^a[a, b] = h^a h = h^2 = v \text{ and } [h, b] = u,$$

we get $H' \geq \langle u, v \rangle = E \cong E_4$, a contradiction.

We have proved that $K_3(G) = [G, G'] = E = G' \cap Z(G) \cong E_4$. Let $a, b \in G - \Phi(G)$ be such that $\langle a, b \rangle = G$. Then $g = [a, b] \in G' - E$, $[g, a] = c_1$, $[g, b] = c_2$, where $\langle c_1, c_2 \rangle = E = K_3(G)$. We set $c_3 = c_1 c_2$ and get

$$[g, ab] = [g, b][g, a]^b = [g, b][g, a] = c_2 c_1 = c_3.$$

We compute the commutator subgroups of our three nonabelian maximal subgroups $X_1 = \langle \Phi(G), a \rangle$, $X_2 = \langle \Phi(G), b \rangle$ and $X_3 = \langle \Phi(G), ab \rangle$, where we note that we must have $|X'_i| = 2$ for $i = 1, 2, 3$.

Since $[g, a] = c_1$ and

$$[a, b^2] = [a, b][a, b]^b = gg^b = g \cdot gc_2 = g^2 c_2,$$

we have $X'_1 = \langle c_1 \rangle$ and so we must have $g^2 c_2 \in \langle c_1 \rangle$. This forces either $g^2 = c_2$ or $g^2 = c_3$.

Since $[g, b] = c_2$ and

$$[a^2, b] = [a, b]^a[a, b] = g^a g = gc_1 \cdot g = g^2 c_1,$$

we have $X'_2 = \langle c_2 \rangle$ and so we must have $g^2 c_1 \in \langle c_2 \rangle$. This forces either $g^2 = c_1$ or $g^2 = c_3$. With the above we get exactly $g^2 = c_3$.

Since $[g, ab] = c_3$ and

$$[a^2, ab] = [a, ab]^a[a, ab] = g^a g = gc_1 \cdot g = g^2 c_1,$$

(where we have used the fact that $[a, ab] = [a, b]$) we have $X'_3 = \langle c_3 \rangle$ and so we must have $g^2 c_1 \in \langle c_3 \rangle$. But we know that $g^2 = c_3$ and so $g^2 c_1 = c_3 c_1 = c_2 \in \langle c_3 \rangle$, a contradiction. We have proved that such 2-groups do not exist!

(ii) Finally, assume that $d(G) = 3$. For any $x, y \in G$ we have $\langle x, y \rangle < G$ and so Proposition 1 implies that $o([x, y]) \leq p$ and $[x, y] \in Z(G)$. But then G' is elementary abelian (of order p^2 or p^3) and $G' \leq Z(G)$ and so we have obtained the groups from part (d) of our theorem. For any $a, b \in G$, $[a^p, b] = [a, b]^p = 1$ and so $\Phi(G) \leq Z(G)$. If $Z(G) \not\leq \Phi(G)$, then there is a maximal subgroup M of G such that $G = \langle M, x \rangle$, where $x \in Z(G)$. But then $G' = M'$ and so $|G'| = 2$, a contradiction. Hence $\Phi(G) = Z(G)$. Theorem 7 is completely proved. \square

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